The Time and Space of Interaction

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Complexity and the λ-calculus

The Simplest Setting: Closed Call-by-Name λ-calculus

We set our analysis in the simplest higher-order Turing-complete framework: pure untyped λ-calculus with weak head reduction. Abstractions are the *only* values.

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Terms	t, u	$::=$	$x \mid \lambda x.t \mid tu$
Context	C	$::=$	$\langle\cdot\rangle \mid Ct$
Reduction	$C\langle(\lambda x.t)u\rangle$	$\to$	$C\langle t\{x \leftarrow u\}\rangle$

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Of course, λ-terms may **diverge**, *e.g.* $\Omega = (\lambda x.xx)(\lambda x.xx) \to \Omega$.

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Can we implement the λ-calculus in a **reasonable** way?

Polynomial Overhead Time Simulations

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A nice proof can be carried out through the use of **abstract machines** (e.g. Krivine's).

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Abstract machines **share** redex arguments, thus consuming SPACE $\simeq$ TIME (unless some garbage collection mechanism is used).

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4. The code never changes.

The Interaction Abstract and Space Complexity

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- **Dal Lago** and **Schöpp**: functional programming in Logspace.
- **Aubert et al.**: logic programming in Logspace.

Positions

The position of the token inside the term t is represented via a pair (u, C) such that $C\langle u\rangle = t$.

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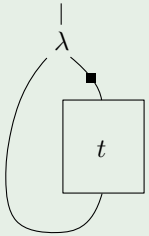
Example



The position of the token ■ in the term $\lambda x.t$ is represented by the position $(t, \lambda x.\langle\cdot\rangle)$.

Traversing β-redexes

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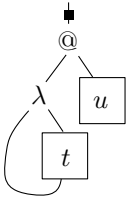

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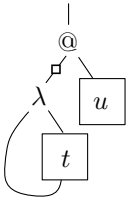

The token ■ moves to the left-hand side of the application, changing color.

$$ru \mid C \mid \blacksquare \quad \rightarrow_{\bullet 1} \quad r \mid C\langle\langle\cdot\rangle u\rangle \mid \square$$

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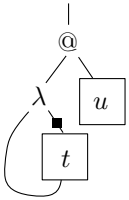

The token $\square$ moves to the body of the abstraction, changing color again.

$$\lambda x.t \;\Big|\; C \;\Big|\; \square \quad \rightarrow_{\bullet 2} \quad t \;\Big|\; C\langle \lambda x.\langle\cdot\rangle\rangle \;\Big|\; \blacksquare$$

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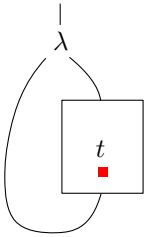

The token ■ is pointing a variable.

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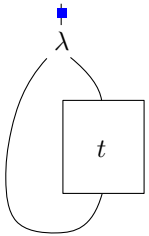

The token ■ moves locally to the binder, changing its mode to blue, *i.e.* the machine is now looking for the argument of the variable.

$$x \mid C\langle \lambda x.D \rangle \mid \blacksquare \quad \rightarrow_{\mathsf{var}} \quad \lambda x.D\langle x \rangle \mid C \mid \blacksquare$$

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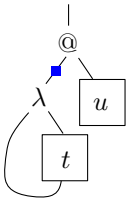

The token ■, *i.e.* looking for an argument is on the left-hand side of an application.

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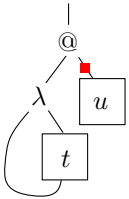

The token moves to the argument, changing its mode to red, *i.e.* looking for the head variable.

$$t \mid C\langle\langle\cdot\rangle u\rangle \mid \blacksquare \quad \rightarrow_{\mathsf{arg}} \quad u \mid C\langle t\langle\cdot\rangle\rangle \mid \blacksquare$$

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Implementation Theorem

The λ-term t has weak head normal form iff the IAM terminates on $(\underline{t}, \langle\cdot\rangle, \epsilon, \epsilon)$.

Types and Complexity

Non-Idempotent Intersection Types

Sequence Types for the CbN λ-Calculus

$$\begin{array}{lrcll} \text{Linear Types} & A & ::= & \star \mid S \to A & \\ \text{Sequence Types} & S & ::= & [A_1 \cdots A_n] & n \geq 0 \end{array}$$

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Correctness and Completeness

$\vdash t : \star$ if and only if $t \Downarrow$.

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- **Relational model**
- Game semantics

Typing Rules

$$\frac{}{x : [A] \vdash x : A}\ \textsf{T-Var} \qquad \frac{\Gamma, x : S \vdash t : A}{\Gamma \vdash \lambda x.t : S \to A}\ \textsf{T-}\lambda \qquad \frac{}{\Gamma \vdash \lambda x.t : \star}\ \textsf{T-}\lambda_\star$$

$$\frac{\Gamma \vdash t : [A'_1, \ldots, A'_n] \to A \quad [\Delta_i \vdash u : A'_i]_{i \in [1,\ldots,n]}}{\Gamma \uplus \sum_{i \in [1,\ldots,n]} \Delta_i \vdash tu : A}\ \textsf{T-@}$$

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The **multiplicative** part is handled by **types**.

The **exponential** part is **no more needed**: in type derivations each subterm has already been copied/discarded. This way, each copy is used only once, in a **linear** way.

The Machine in Action

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : [[\star] \to \star] \vdash x : [\star_{\downarrow 16}] \to \star_{\uparrow 6}} \quad \dfrac{}{y : [\star] \vdash y : \star_{\uparrow 17}}}{y : [\star], x : [[\star] \to \star] \vdash xy : \star_{\uparrow 5}}}{y : [\star] \vdash \lambda x.xy : [[\star_{\uparrow 15}] \to \star_{\downarrow 7}] \to \star_{\uparrow 4}}}{\vdash \lambda y.\lambda x.xy : [\star_{\downarrow 18}] \to [[\star_{\uparrow 14}] \to \star_{\downarrow 8}] \to \star_{\uparrow 3}} \quad \dfrac{}{\vdash \mathsf{I} : \star_{\uparrow 19}} \quad \dfrac{\dfrac{}{z : [\star] \vdash z : \star_{\uparrow 11}}}{\vdash \lambda z.z : [\star_{\downarrow 12}] \to \star_{\uparrow 10}}}{\vdash (\lambda y.\lambda x.xy)\mathsf{I} : [[\star_{\uparrow 13}] \to \star_{\downarrow 9}] \to \star_{\uparrow 2}}}{\vdash (\lambda y.\lambda x.xy)\mathsf{I}(\lambda z.z) : \star_{\uparrow 1}}$$

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2. run time = number of states = number of occurrences of $\star$ in the derivation.

Counting the $\star$: a Semantic Measure of the IAM run time

$$\|\star\| = 1 \qquad \|S \to A\| = \|S\| + \|A\| \qquad \big\|[A_1, \ldots, A_n]\big\| = \textstyle\sum_{1 \le i \le n} \|A_i\|$$

$$\frac{}{\vdash^{0} \lambda x.t : \star} \qquad \frac{\vdash^{w}}{\vdash^{w+\|S \to A\|} \lambda x.t : S \to A}$$

$$\frac{}{\vdash^{\|A\|} x : A} \qquad \frac{\vdash^{w} \quad \left[\vdash^{v_i}\right]_{i \in [1,\ldots,n]}}{\vdash^{w+\sum v_i+\|A\|} tu : A}$$

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Ultimately, IAM inefficiency comes from the use of **higher-order** types.

Recap

- IAM run time can be characterized by intersection type derivations.

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Space Type Measure

$$\|\star\| := 0 \qquad \|T \to A\| := \max\{\|T\|, \|A\| + 1\}$$

$$\left\|[G_1, .., G_n]\right\| := \mathsf{X} + \max_i\{\|G_i\|\}$$

The Inference Rules

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The Inference Rules - Weighted

$$\frac{}{x : [A] \overset{\|A\|}{\vdash} x : A}\ \text{T-Var}$$

$$\frac{\Gamma, x : T \overset{w}{\vdash} t : A}{\Gamma \overset{\max\{w, \|T \to A\|\}}{\vdash} \lambda x.t : T \to A}\ \text{T-}\lambda$$

$$\frac{\Gamma \overset{w}{\vdash} t : T \to A \quad \Delta \overset{v}{\vdash} u : T}{\Gamma \uplus \Delta \overset{\max\{w,v\}}{\vdash} tu : A}\ \text{T-@}$$

$$\frac{}{\Gamma \overset{0}{\vdash} \lambda x.t : \star}\ \text{T-}\lambda_\star$$

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Recap

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Lemma

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$\|\mathbb{X}_n\| = \Omega(2^n)$.

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- Given a TM that halts in n steps, its encoding in the λ-calculus consumes $\Omega(n)$ space when evaluated on the IAM.

Ongoing and Future Work

Analyzing the space/time behaviour of environment machines *with* garbage collection.