Homework: Prove the Exhaustible State Invariant

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1 Class Definitions

Lemma 1.1 (Lifting)**.** *If* $(t, C, L, T, d) \to^n_{\text{IAM}} (u, D, L', T', d')$*, then* $(t, C, L, T \cdot T'', d) \to^n_{\text{IAM}} (u, D, L', T' \cdot T'', d')$*.*

Definition 1.2 (Tape tests)**.** *Let* $q = (t, C, L, T' \cdot l \cdot T'', d)$ *be a state. Then the* tape test *of* q *of focus* l *is the state* $q_l = (t, C, L, T' \cdot l, \downarrow)$*.*

Definition 1.3 (State surrounding a position)**.** *Let* $l = (t, D, L')$ *be a logged position. A state* q *surrounds* l *if* $q = (t, \underline{C_n\langle D\rangle}, L' \cdot L_n, \epsilon)$ *for some context* C_n *and log* L_n*.*

Definition 1.4 (Exhaustible States)**.** $\mathcal{E}$ *is the smallest set of states* q *such that if* q_l *is a tape test of* q*, then* $q_l \to^+_{\text{IAM}} q''$*, where* q'' *surrounds* l*. States in* $\mathcal{E}$ *are called* exhaustible*.*

Proposition 1.5 (Exhaustible invariant)**.** *Let* q *be a IAM reachable state. Then* q *is exhaustible.*

Corollary 1.6 (Logged Positions Never Block the IAM)**.** *Let* q *be a reachable state. If* $q = (\underline{\lambda x.D\langle x\rangle}, C, L, l \cdot T)$ *then* q *is not final.*

Proof. By the exhaustible invariant (Prop. 1.5), q is exhaustible. Then, its tape test $q_l := (\underline{\lambda x.D\langle x\rangle}, C, L, l)$ does at least one transition towards a state $q'_l \neq q_l$. Note that q_l is q with empty tape. Now, we conclude by lifting this transition by using the tape lifting lemma (Lemma 1.1). □

Log Tests and Position Changes. To define the log test focussing on the m-th logged position l_m in the log of a state $(t, C_n, l_n \cdots l_2 \cdot l_1, T, d)$, we remove the prefix $l_n \cdots l_{m+1}$ (if any), and move the current position up by $n - m$ levels. Moreover, the tape is emptied and the direction is set to $\uparrow$. Let us define the position change.

Let (u, C_{n+1}) be a position. Then, for every decomposition of n into two natural numbers m, k with $m + k = n$, we can find contexts C_m and C_k, and a term r satisfying exactly the following condition: $t = C_m\langle rC_k\langle u\rangle\rangle$. Then, the $m+1$-*outer context* of the position (u, C_{n+1}) is the context $O_{m+1} := C_m\langle r\langle\cdot\rangle\rangle$ of level $m + 1$ and the $m + 1$-*outer position* is $(C_k\langle u\rangle, O_{m+1})$.

Note that the m-outer context and the m-outer position (of a given position) have level m. It is easy to realize that any position having level n has *unique* m-outer context and m-outer position, for every $1 \leq m \leq n+1$, and that, moreover, outer positions are hereditary, in the following sense: the i-outer position of the m-outer position of (u, C_{n+1}) is exactly the i-outer position of (u, C_{n+1}).

Definition 1.7 (Log tests)**.** *Let* $q = (t, C_n, l_n \cdots l_2 \cdot l_1, T, d)$ *be a state with* $1 \leq m \leq n$*, and* (u, O_m) *be the* m*-outer position of* (t, C_n)*. The* m-log test *of* q *of focus* l_m *is the state* $q_{l_m} := (u, O_m, l_m \cdots l_2 \cdot l_1, \epsilon, \uparrow)$*.*

By definition, log tests for q do not depend on the direction of q, nor on the underlying tape, and they are stable by head translations of the position (t, C_n) of q, in the sense that if $t = H\langle r\rangle$ then $q = (t, C_n, L, T, d)$ and its head translation $(r, C_n\langle H\rangle, L, T', d)$ induce the same log tests (because the two positions have the same outer positions and the two states have the same logs). Remember that head contexts H are nothing but contexts C_0 of level 0.

Lemma 1.8 (Invariance properties of log tests)**.** *Let* $q = (t, C_n, L_n, T, d)$ *be a state. Then:*

1. Direction*: the dual* (t, C_n, L_n, T, d^1) *of* q *induces the same log tests;*
2. Tape*: the state* (t, C_n, L_n, T', d) *obtained from* q *replacing* T *with an arbitrary tape* T' *induces the same log tests;*
3. Head translation*: if* $t = H\langle r\rangle$ *then the head translation* $(r, C_n\langle H\rangle, L_n, T', d)$ *of* q *induces the same log tests.*
4. Inclusion*: if* $C_n = C_m\langle C_i\rangle$ *and* $L_n = L_i \cdot L_m$ *then the log tests of* $(C_i\langle t\rangle, C_m, L_m, T', d)$ *are log tests of* q*.*

2 Questions

- Try to prove Prop. 1.5. You should find a problem already in the case $\rightarrow_{\bullet 1}$. In order for the proof to go through you need to refine Def. 1.2.
- Proceed with the proof, you should find a problem with $\rightarrow_{\mathsf{bt1}}$. Something should be said also about the logged positions on the log! Do you remember we defined log tests in class (there was a reason)? The definition of exhaustible state should be strengthened using also log test.
- At this point $\rightarrow_{\mathsf{bt2}}$ does not work anymore. We should strengthen the invariant even more! How? Think about the reducibility technique.
- There is still a subtlety. Can you spot it?