Homework: Theories and Trees

March 28, 2024

1 $\mathcal{H}^*\eta = \mathcal{H}^*$

Definition 1.1.

- $\lambda\eta$ is the least λ-theory containing $=_\eta$.
- $\mathcal{H}^*\eta$ is the least λ-theory containing $\mathcal{H}^*$ and $\lambda\eta$.

Definition 1.2. A term t is $\beta\eta$ solvable if and only if there exists a context H such that $H\langle t\rangle \to^*_{\beta\eta} \mathsf{I}$.

Theorem 1.3. *$\beta\eta$ reduction is confluent.*

Lemma 1.4. *A term t is solvable if and only if it $\beta\eta$ solvable.*

Considering all the facts above, prove that:

Theorem 1.5. *$\mathcal{H}^*\eta = \mathcal{H}^*$. Hint: consider the different cases for which two terms are equated in $\mathcal{H}^*\eta$.*

2 Algebraic properties of approximants

Prove the following considering the equivalence of definitions of $\mathcal{A}(t)$ in Giulio's slides for granted. Beware, in Giulio's slides what we called $\leq_\bot$ is written $\sqsubseteq$, and $\mathsf{da}(t)$ is written $\omega(t)$.

Lemma 2.1. *The set $\mathcal{A}(t)$ is an ideal w.r.t. $\leq_\bot$:*

1. *$\bot \in \mathcal{A}(t)$.*
2. *If $r, u \in \mathcal{A}(t)$, then $r \sqcup u \in \mathcal{A}(t)$.*
3. *Downward closed: $u \leq_\bot r \in \mathcal{A}(t)$ implies $u \in \mathcal{A}(t)$.*

Bonus exercise, prove the following:

Proposition 2.2. *The two definitions of $\mathcal{A}(t)$ are equivalent.*